

Bolzano against Kant's Pure Intuition

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Abstract The 19th-century polymath Bernard Bolzano is widely regarded as a figure who decisively showed that even in Kant's time, there were philosophical and mathematical reasons to think that an intuition-based philosophy of mathematics was untenable. This paper identifies and clarifies two of Bolzano's longstanding objections to Kant's philosophy of mathematics. Once this is done, I defend Kant from these objections. To conclude, I discuss a deeper disagreement between Bolzano and Kant's philosophical approaches. Namely, Bolzano rejected that a "critique of reason" (i.e., an examination of the principles and limits of human cognition) was necessary in order to secure and explain reliable forms of inference and claims to a priori knowledge.

1 Introduction

In *The Semantic Tradition from Kant to Carnap*, Alberto Coffa praises the 19th-century mathematician and philosopher Bernard Bolzano for founding a distinctively anti-Kantian "semantic" tradition, which Coffa defines as follows:

The semantic tradition may be defined by its problem, its enemy, its goal, and its strategy. Its problem was the a priori; its enemy, Kant's pure intuition; its purpose, to develop a conception of the a priori on which pure intuition played no role; its strategy, to base that theory on a development of semantics. (Coffa 1991, p. 22)

In terms that parallel Coffa's appraisal, Bolzano saw himself as taking up the task of showing how the "path ... which the *Critique* pursues ... is not at all the right one" (Příhonský 1850, p.146). According to Bolzano, in order to provide a philosophically satisfying account of the nature and possibility of a priori knowledge, one should begin with a careful examination into "nature of the truths in

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themselves” (Ibid., p.146) rather than analyzing our cognitive capacities, as Kant mistakenly does. In this paper, I seek to identify what Bolzano found to be problematic about Kant’s intuition-based approach to mathematical knowledge. This will not only put us in a better position to determine the resilience of Kant’s views against Bolzano’s criticisms, but it will allow us to appreciate the philosophical problems that Bolzano’s own views seek to overcome.

The effect of Kant’s critical philosophy on Bolzano’s thought cannot be underestimated as Bolzano held a longstanding philosophical engagement with Kant’s critical philosophy and Kantian philosophers such as Schultz and Kiesewetter. In fact, Bolzano’s influential 1810 treatise, *Beträge zu einer begründeteren Darstellung der Mathematik*¹ (*BD*), includes an appendix dedicated to arguing why Kant’s notion of *pure intuition* was both incoherent and unable to ground mathematical knowledge. Bolzano continues his quarrel with Kant in his 1837 magnum opus, the *Wissenschaftslehre*² (*WL*). In addition to presenting global criticisms of Kant’s critical philosophy, Bolzano’s *WL* seeks to provide viable alternatives to various Kantian positions. Following the publication of *WL*, Bolzano’s philosophical engagement with Kant culminates through the writings of Bolzano’s pupil František Příhonský. With the consultation and approval of Bolzano, Příhonský, in 1850, published *Neuer Anti-Kant*³ (*NAK*), a systematic commentary on Kant’s first *Critique* from the point of view of Bolzano’s *WL*. As the title suggests, Příhonský’s *NAK* was far from a positive assessment of Kant’s first *Critique*.⁴

Despite Bolzano’s disagreements with Kant, it is important to take note of some of the illuminating similarities between the respective overarching philosophical projects of these two thinkers. Just as Kant thought that philosophy was stuck in a “battlefield of [...] endless controversies” (Kant 1781, Aviii), Bolzano thought errors plaguing the sciences were responsible for why “in mathematics one also finds long disputes over certain subjects, e.g., the measure of vis viva, the ratio of force to resistance with wedges, the distribution of pressure over several points, etc., which are still unresolved” (Bolzano 1837, §315). While Kant sought to resolve disputes in metaphysics by distinguishing the method of philosophy from that of mathematics, Bolzano sought to resolve disputes in the sciences by instigating a program of foundational reform that critiqued the correctness of and standards for the definitions, axioms, and proofs used in the sciences, especially those in mathematics. In order to preserve the status of the sciences as such, Bolzano thought their presentation had to reflect objective ground-consequence relations between something like mind-independent propositional contents.

¹ “Contributions to a better-grounded Presentation of Mathematics.”

² “Theory of Science.”

³ “New Anti-Kant.”

⁴ More specifically, *NAK* was instigated by Bolzano in 1837, the same year *WL* was published. Příhonský was responsible for writing and publishing *NAK*, and Bolzano was actively involved in its development, providing Příhonský with input and eventually, his approval of the finished product. Thus, there is good reason to think that *NAK* reflects Bolzano’s own views.

Like other critics of Kant at the time, Bolzano took issue with Kant's claim that constructions in pure intuition could ground mathematical knowledge.⁵ This paper focuses on two closely related objections of Bolzano's, both of which reflect critical themes recurring throughout his works:

CAT: By its nature, intuition cannot account for the generality of mathematical cognition or the necessity of mathematical inference.⁶

DIS: Even if intuition can help discover that a mathematical proposition is true, intuition plays a dispensable role in securing mathematical knowledge and inference.

For convenience, I'll call the first objection, the *categorical objection*, or CAT, and the second, the *dispensability objection*, or DIS. The purpose of this paper is to clarify these two objections from the point of view of Bolzano's corpus. In order to properly assess the force of these objections, it will be important to distinguish the various roles that Kant claims intuition to play in mathematical cognition and inference. Once these objections are clarified, my plan is to defend Kant against them. Thus, while Coffa (1991), Rusnock (1999), Lapointe (2011), and others champion Bolzano as decisively showing that Kant's views on mathematics were untenable even in Kant's time, I argue that neither the categorical nor the indispensability objection do the job. To conclude, I briefly discuss what I take to be fundamentally driving Bolzano's dissatisfaction with Kant's view: Bolzano rejected that a "critique of reason" was necessary to explain and secure reliable forms of synthetic inference and claims to a priori knowledge.

2 Kant's Theory of Cognition

Let's begin with a brief overview of the basics behind Kant's theory of cognition as explicated in the first *Critique*. For Kant, cognition (i.e., "Erkenntnis") is a technical term used to refer to one's awareness of an object or objects as having determinate properties. As such, cognition is a type of epistemic achievement that results from

⁵ The critics I have in mind are those who ascribe to what might be suitably called a "Wolffian" or "Leibnizian" philosophy of mathematics according to which a mathematical proof of a proposition P (which is necessary and sufficient to furnish mathematical knowledge of P) proceeds and "depends on the principle of contradiction and in general on the doctrine of syllogisms" (Lambert 1761, §92.14). See Heis (2014) for more on this debate between the Wolffian and the Kantian.

Secondly, it is important to point out that, according to Bolzano, there is a distinction between the actual reasons why a proposition P is true and what merely convinces us that P is true. The former tracks objective ground-consequence relations mentioned in the previous paragraph. These relations hold regardless of whether there are any minds or epistemic agents.

Bolzano was not the first to make this distinction. In fact, the distinction between the metaphysical ordering of truths and the doxastic order by which one comes to believe a truth can be traced back to Aristotle (1993) in Book I of the *Posterior Analytics*.

⁶ In this paper, I understand a proposition P to be "general" or "universal" if P applies to a class of objects such as all triangles or all natural numbers and not just particular instances. I take "necessity" to mean something similar except that it is a property that applies primarily to arguments and inferences.

the interplay between two distinct faculties of representation, namely, sensibility and understanding. Through sensibility we are affected passively by the world, receiving what Kant calls *intuitions*. Intuitions are particular representations that refer immediately to an object. Contrasting with the passive faculty of sensibility, understanding is an active faculty responsible for unifying representations into *concepts, thoughts, and judgments*. Concepts are general (i.e., repeatably applicable) representations that relate to an object mediately through marks.⁷

Since mathematical cognitions are arguably non-empirical (i.e., they are not about anything, nor can they be justified by anything one encounters in experience), the story becomes slightly more complicated. As Kant argues, instances of the concepts of mathematics are not given in experience. Rather, to grasp a mathematical concept for Kant is *to be able to construct it* in pure intuition according to a general procedure given by the concept's construction schema, which is broadly understood to be something like its genetic definition.⁸ Kant argues in the *Transcendental Aesthetic* that space and time are a priori intuitions. As such, they constitute the forms of sensibility; i.e., they are precisely those invariant and necessary features of experience and how objects are given to us.

To give an example of what Kant has in mind when he speaks of construction in pure intuition, Kant would argue that in order to grasp the concept <triangle>, one would have to construct a triangle in pure intuition according to its construction schema. From there, one can reason with one's construction and prove that, for instance, the sum of its angles is two right angles. Similarly, an intuition of a homogeneous unit constructed in the pure intuition of time underwrites the concept of quantity as such needed to reason in arithmetic or algebra. From a Kantian perspective, when one begins a proof with the phrase "let n be an arbitrary natural number," one is reasoning with a constructed homogeneous unit.

I argue that constructions in pure intuition are meant to play at least two important roles in Kant's account of mathematical cognition:

- AX: Constructions are needed in order to have cognition of axioms as self-evident truths that are general in nature.
- INF: Constructions are needed to justifiably make necessary and general inferences that go beyond what can be inferred by means of concept-containment relations (i.e., conceptual analysis) or traditional logic (i.e., an Aristotelian term logic with additional principles regarding disjunctive and hypothetical judgments).

⁷ To give a mundane mathematical example, the concept <triangle> (I will use angle brackets to denote concepts) has among its marks the concept <three-sided>.

⁸ Classically understood, genetic definitions are definitions that reveal the "genesis" or "cause" of the thing being defined. What this means is often spelled out in terms of a finite procedure that specifies how to construct the object being defined presumably from a given or privileged set of entities. For instance, a genetic definition for a triangle provides a procedure of constructing a triangle from more basic kinds of objects (e.g., points and lines) by certain rules and methods of (perhaps, straightedge and compass) construction. No claim here is made as to whether for Kant, genetic definitions have to be *unique*. Friedman (2010) construes schemata as *functions* of a certain type. They are functions whose inputs (e.g., three arbitrary lines) are a given set of entities and whose output is a representation (e.g., a triangle constructed from those three lines). Precising "schemata" in terms of genetic definitions or functions (in Friedman's sense) works equally well for the purposes of the paper.

Before proceeding to clarify what these two roles consist in, it is important to note that the essence of Bolzano's categorial and dispensability objection is to undercut AX and INF. The categorial objection doubts whether intuition can play either of these roles, while the latter maintains that intuition—even if it has any non-trivial epistemic role at all—is entirely indispensable in acquiring mathematical knowledge.

According to Kant, axioms are *synthetic* truths that are immediately evident and unprovable.⁹ As synthetic, axioms involve concepts that do not contain one another.¹⁰ Famously, Kant argues that the axioms of Euclidean geometry are genuine paradigm cases of axioms according to his understanding of the word. Thus, in particular, the axiom asserting that “there exists a line between any two points” is *immediately known* once one constructs two points in pure intuition.

Since axioms involve concepts that do not bear the proper requisite concept-containment relations, Kant argues that one cannot arrive at cognition of axioms through conceptual analysis. Without any further epistemic resources, there is nothing that privileges a genuine axiom from arbitrarily conjoined concepts since a mere appeal to the concepts involved will not do the trick. Thus, Kant argues that not only can intuitions deliver genuine mathematical knowledge, they are *needed* to secure knowledge of axioms (i.e., AX).¹¹ And so Kant's pure intuition is what separates a genuine axiom from arbitrarily conjoined concepts. As a consequence of Kant's view, the cognition of the Euclidean axiom asserting that “between two points there exists a line” results immediately from constructing two points in pure intuition.

Kant contrasts axioms with judgments such as “the sum of the angles of a triangle is equal to two right angles,” which are inferred from premises. According to Kant, the difference between the two is straightforward. Cognition that is *inferred* requires an *inference* and a set of premises from which the cognition is *inferred*. On the other hand, immediate cognition does involve either premises from which an inference is made.

Kant argues that constructions are not only necessary for explaining our cognition of axioms, i.e., he argues for AX, but he argues that they are needed to draw reliable inferences that go beyond the rules of general logic, i.e., he argues for INF.

According to Kant, general logic concerns the “logical form in the relation of cognitions to one another” (Kant 1781, B76-79). As such, general logic consists in the rules that hold of any subject matter regardless of whether there are any epistemic agents at all. Based on this characterization of general logic, Kant concludes

⁹ See, for example, Kant (1781, A303) and (Ibid., A732/B760) for clear articulations of Kant's conception of axioms, especially as they compare with knowledge that is *inferred*.

¹⁰ Kant formulates the distinction between analytic and synthetic judgments in different possibly inequivalent ways. Nonetheless, in this paper, the distinction will be understood in terms of concept-containment relations. As already touched on, this distinction hinges on a certain view of concepts, namely, that concepts have *marks*, which are just further concepts, and that concepts admit of decomposition in terms of these marks. This view of concepts was somewhat standard and shared among thinkers such as Lambert and even Bolzano.

¹¹ That intuition *can* furnish knowledge of axioms is taken to be evident for Kant in light of the epistemic legitimacy of Euclidean diagrammatic geometry. See Friedman (2010) for more on the relationship between Kant and Euclidean geometry.

that the principles of logic were exhausted by traditional logic. Regardless of what one takes as constituting the principles of logic, what matters for Kant is that they concern the mere form of thoughts.

To return to the above example (i.e., that the sum of the angles in any triangle is two right angles), Kant argues that it is not possible for us to provide a proof that proceeds solely by means of general logic and conceptual analysis. One needs to begin with an a priori construction of a triangle, and perform auxiliary constructions until one arrives at the theorem under question.

To summarize, Kant's philosophy of mathematics is primarily concerned with mathematical cognition as an epistemically special mental state that arises from the interplay between the faculties of sensibility and understanding. Logical and conceptual resources (i.e., general logic and concept-containment relations) are unable to account for (1) our knowledge of mathematical axioms as *immediate* and *synthetic*; or for (2) our knowledge of inferred extra-formal mathematical knowledge. Since logical or conceptual means won't do the trick, a priori constructions, according to Kant, account for and are needed to account for (1) and (2).

3 The Categorical Objection

At this point, one may already anticipate Bolzano's objections. Let's first focus on the categorical objection, which is best expressed in the following passage from Bolzano's *BD*:

How do we come, from the intuition of that single object, to the feeling that what we observe in it also belongs to every other one? Through that which is single and individual in this object, or through that which is general? Obviously only through the latter, i.e., through the concept, not through the intuition. (Bolzano 1810, A.7)

According to Bolzano, Kant's philosophy of mathematics falls short since Kant has yet to provide a compelling account of how intuitions, as essentially singular, can underwrite our knowledge of mathematical truths and inference as being in any way necessary. According to Bolzano, one can only be aware of the necessity and generality of mathematical truths if one *proceeds from concepts* rather than intuitions. It seems that intuitions *by their very nature* are unable to show that a mathematical truth *has* to be true or that an inference is *valid* since intuitions are merely instantiations of concepts, and inferring from particulars to general statements is in general unreliable.

In order to defend Kant from the categorical objection, one must show how particular intuitions can reliably guide an epistemic agent to cognition of general mathematical truths. As the following passage illustrates, Kant's solution is to appeal to the generality of the procedure or schema by which a concept is constructed in pure intuition:

No image at all would ever be adequate to the concept of a triangle in general. For it would never attain the universality of the concept, which makes it hold for all triangles, whether

right-angled, acute-angled, and so on, but would always be limited to only a part of this sphere. The schema of the triangle can never exist anywhere but in thought, and it signifies a rule of synthesis of the imagination with respect to pure figures in space. (Kant 1781, A140/B180)

According to Kant, since constructions proceed through schemata, whatever is known from using a construction does not depend on any of the particular determinations and magnitudes that end up being constructed in pure intuition. Thus, as Kant notes, through schemata one is able to “consider the universal in the particular” (Kant 1781, A714/B742).

Bolzano was well aware of Kant's appeal to schemata as a possible reply to the categorial objection. However, Bolzano remained unsatisfied:

If the schema is nothing but an idea of the way in which a circle is generated, i.e., nothing but the well-known definition of a circle, which is called a genetic definition, namely, the concept of a line which is described by a point that moves in a plane in such a way that it always maintains the same distance from a given point. Thus if one recognizes the truth of a synthetic proposition through the consideration of the schema of its subject-concept, one recognizes this truth from the consideration of mere concepts. (Bolzano 1837, §305)

Ironically, an appeal to schemata seems to support Bolzano's claim that only through conceptual means can one preserve the universality of mathematical truths; an inference from a genetic definition is just an inference from concepts.

In a sense, Kant *agrees* with Bolzano. The generality of mathematical cognition and the necessity of an inference in mathematics is due to the general nature of concepts and their schemata. However, synthetic mathematical cognition requires that the mathematical concepts involved be constructed in pure intuition. While one may have a rule to draw triangles, one needs to actually carry out the construction in pure intuition in order to cognize mathematical axioms and make inferences that go beyond concept containment relationships and traditional logic. As Kant notes, one can “reflect on a [mathematical] concept as long as he wants, yet he will never produce anything new” (Kant 1781, A715/B744). One must construct the concept as “through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and *at the same time* general solution to the question” (Kant 1781, A717/B745, my emphasis). To succinctly make the point, while concepts and schemata are *necessary* for mathematical cognition, they are not sufficient.

To recap, throughout his writings, Bolzano argues that intuition is not the type of thing that could ground mathematical cognition, immediate or inferred. I hope to have shown that the particularity of intuitions does not preclude them from enjoying a necessary role in mathematical cognition.

4 The Dispensability Objection

While Bolzano's categorial objection questions whether intuitions are even the right kind of thing that can underwrite mathematical cognition, the dispensability

objection attacks Kant's claim that intuitions are *necessary* for mathematical cognition, immediate or inferred. In fact, Bolzano argues that intuition (a priori or otherwise) is largely inessential to mathematical knowledge.

Bolzano argues for the dispensability objection in at least two ways. By providing counterexamples, and by considering some of the consequences of his theory of propositions, the most mature formulation of which is developed in the *WL*. In what follows, I discuss each argument independently.

4.1 Bolzano's Counterexamples

In *BD*, Bolzano provides a series of examples where intuition appears to play an inessential role in securing mathematical knowledge. For example, as long as one assumes the associativity of addition on the natural numbers, the pure intuition of time is not needed to explain one's knowledge of arithmetical sums. Due to associativity, "one looks at the set of terms, not their order" (Bolzano 1810, A.8). One can then compute " $7 + 2$ " by defining the natural numbers and addition recursively, i.e., by assuming that an undefined primitive term " 1 " denotes a number and what results from repeated applications of the successor function to " 1 " also denotes a number. Thus, one can compute " $7 + 2$ " in the following way without the aid of intuition:

$$\begin{array}{ll}
 7 + 2 = 7 + (1 + 1) & \text{(By definition)} \\
 = (7 + 1) + 1 & \text{(By associativity)} \\
 = 8 + 1 & \text{(By definition)} \\
 = 9 & \text{(By definition)} \\
 = 9 & \text{(by substitution)} \\
 & \text{QED}
 \end{array}$$

It is not difficult to defend Kant from this purported counterexample. One can of course define and stipulate whatever one likes. Kant would argue that one can only know the correctness of such definitions and rules of inference (the latter of which are either axioms or proven from axioms) if one verifies them in pure intuition. According to Kant, because arithmetic sums like " $7 + 1 = 8$ " cannot be known from pure conceptual analysis, help must be sought in pure intuition. Without such verification, Bolzano's purported logical proof that " $7 + 2 = 9$ " fails to exhibit the synthetic nature of arithmetical cognition that Bolzano, like Kant, attributes to computing sums.¹²

¹² It might be helpful to recall Russell's criticism of "the method postulating" by which one postulates mathematical axioms and definitions without verifying their truth and consistency. According to Russell, without doing the hard philosophical and mathematical work of verifying the axioms and definitions that one has laid down, the method of postulation amounts to nothing more than "theft over honest toil" (Russell 1919, p.71). Thus, Bolzano cannot appeal to stipulated definitions

Furthermore, given that Bolzano in *BD* adopts Kant's formulation of the distinction between the analytic and synthetic, Bolzano would be rationally bound to agree with Kant that the associativity of addition on the natural numbers is a *synthetic truth* (Bolzano 1810, §17).¹³

Before proceeding, it is important to clear up a possible misunderstanding on Bolzano's end. Regardless of whether associativity is analytic or synthetic, one should note that the pure intuition of time does not ground arithmetic by virtue of the fact that we do computations in time. We certainly draw syllogistic inferences in time, and yet, such inferences are not grounded in the pure intuition of time. The pure intuition of time is the material with which one can construct homogeneous units needed to perform computations.¹⁴ Thus, if the associativity of addition is not analytic for Kant, then our cognition of the associativity of sum is only possible if we construct numerical units in pure intuition.

4.1.1 A Counterexample from Analytic Geometry

In a similar spirit to the above purported counterexample, Příhonský in *NAK* argues that the rise of analytic geometry has shown that constructions in pure intuition are inessential even in the context of geometry:

Rather [than relying on intuition, it] is possible to deduce all truths of geometry from the correct definition of space, without ever allowing even once a conclusion that has no other ground for justification in favour of it besides what visual inspection teaches. The well-known analytical geometry provides us with examples of this procedure that show at least how many geometrical truths which visual inspection teaches can also be brought out without any appeal to it, through mere inferences. (Příhonský 1850, p. 116)

According to Příhonský, one can solve problems in geometry by positing systems of equations and a coordinate system without any appeal to visualization or diagrams. Hence, instead of proceeding through visualization, and hence, intuition, Příhonský claims that one can just infer from these algebraic tools and the "correct definition of space" in order to arrive at geometric truths.

In order to have cognition of claims about magnitudes as such (the subject matter of algebra for Kant), one needs to construct magnitudes in pure intuition. Again, for Kant, a priori construction is meant to account for the immediate certainty of algebraic axioms, and for drawing synthetic inferences. Thus, an appeal to analytic geometry does not necessarily show that intuition is inessential for mathematical knowledge.

and axioms in order to undermine Kant's philosophy of mathematics since the validity of such definitions and axioms are the very thing intuitions are supposed to secure, according to Kant. I thank one of the anonymous referees for pointing this out to me.

¹³ This is admittedly a delicate issue since Bolzano's conception of analyticity and syntheticity evolved throughout his writings. As noted, Bolzano in *BD* defines "analytic" in terms of concept-containment, and defines "synthetic" as being not analytic. However, in *WL*, Bolzano defines "analytic" according to a variable term criterion. See Chapter 5 of Lapointe (2011) for more on Bolzano's mature conception of analyticity.

¹⁴ See Kant (1781, B182-3).

In sum, as I have argued, it is difficult to see how an appeal to the associativity of addition on the natural numbers nor analytic geometry can establish that intuition is eliminable from mathematical knowledge. And in general, any purported counterexample that leaves unexplained the “correctness” of certain definitions or (synthetic) forms of inference fails to demonstrate that intuition is dispensable from mathematical knowledge since intuition is the very thing that witnesses such correctness, according to Kant.

4.2 Bolzano’s Theory of Propositions

In the previous section, I argued that neither Bolzano or Příhonský were able to generate convincingly genuine counterexamples to Kant’s account of mathematical knowledge. In this section, I look at an alternative strategy of undermining Kant’s claim that intuition plays an indispensable role in securing mathematical knowledge. This strategy consists in putting forward an account of mathematical cognition whereby mathematical cognition (of the sort Kant describes) can be secured by purely conceptual means. Such a strategy is pursued by Bolzano extensively in *WL* by way of his theory of propositions, which will be discussed shortly. However, I will first sketch out Bolzano’s argument for the dispensability of intuition that falls out of his theory of propositions.

According to Bolzano, the propositions of mathematics are of a certain type, namely, they are *conceptual truths*; they are truths the constituents of which consist solely of concepts. Bolzano argues that the ground of a conceptual truth can only be further conceptual truths, and the case is no different in mathematics. Even if intuition can help guide an epistemic agent to mathematical knowledge, its genuine ground is not found in intuition but in a further conceptual truth. Thus, for Bolzano, intuition is entirely dispensable.

To better understand this argument for the dispensability objection, it is important to note that Bolzano not only questioned the coherence of Kant’s philosophy of mathematics, but he questioned the fundamental notions on which it was built. Bolzano found Kant’s formulation of terms such as “intuition,” “concept,” “a priori,” “a posteriori,” “necessity,” and “generality” problematic because Kant defined such terms with reference to our cognitive capacities. Indeed, in the Appendix to *BD*, Bolzano argues that the notion of “a priori intuition” is hopelessly obscure because Kant is working with problematic notions of “a priori” and “a posteriori.” For Kant, the distinction between a priori and a posteriori cognitions is based on how a given cognition arises (i.e., whether it arises from experience or not). However, under this conception, judgments are not intrinsically a priori or a posteriori. For instance, Kant’s definitions allow one to have empirical cognition of the proposition that “all bachelors are unmarried” by observing a group of bachelors in a room and asking each of them whether he is a bachelor or not. However, because the concept <unmarried> is contained in the concept <bachelor>, one can also have an a priori cognition of the same proposition.

For Kant, then, whether a proposition is a priori or not depends on how the proposition is cognized. In Bolzano's eyes, the problem with Kant's definitions is that they do not pick out *intrinsic* properties of propositions but rather *subject-relative* ones. If the highest kind of mathematical knowledge consists in knowing a mathematical proof that reflects objective relations of ground and consequence and in knowing that it is a grounding proof, then Kant's subject-relative definitions simply won't do. Knowledge of mathematics should not focus on the psychological story on how cognitions are formed. Rather, an account of mathematical knowledge should primarily concern itself with objective, that is, mind-independent ground-consequence relations that obtain between propositions. Bolzano and Přihonský's dissatisfaction with Kant's definitions is captured in the following passage:

It almost seems as if [Kant] had wanted to make the determination of whether a cognition is *a priori* or *a posteriori* dependent upon the merely contingent circumstance of how we gain the cognition, whether in the course of experience or through mere reflection. But how unsteady is such a distinction! How uncertain when it comes to making a decision on the question of whether a cognition belongs to one or the other of the two classes! Must one not admit that we reach most, and to a certain extent, all of our cognitions, even the *a priori* cognitions, by means of experience? (Přihonský 1850, p. 47)

At the same time, this is not to say that Bolzano was not concerned with the epistemic agent. In fact, it is the epistemic agent's duty to discover the genuine grounds of a proof rather than satisfy herself with intuitions that merely confer subjective confidence in a proposition's being true.

According to Bolzano, mathematical cognition and representational entities are better understood in relation to what he calls *propositions in themselves* (i.e., "Satz an Sich"), which are the objective mind-independent contents or intentions of linguistic assertions. If there is no ambiguity, I will call "propositions in themselves" simply "propositions." According to Bolzano, the proposition expressed by a linguistic assertion differs from its referent, which is something like a state of affairs.

Representations, which Bolzano calls "ideas in themselves," are the components of propositions. And like propositions, representations are mind-independent entities that differ from referents. Bolzano defines intuitions as simple and singular representations. For a representation to be simple means that it cannot be broken down into further component representations. To be singular means that the representation has only one referent. Concepts are defined to be representations that are not intuitions and have no intuitions as parts.

Bolzano's distinction between a priori and a posteriori falls out of the above definitions. According to Bolzano, an *a priori proposition* is one whose components are concepts. In contrast, *a posteriori propositions* are those which contain an intuition. Naturally, Bolzano refers to a priori propositions as *conceptual propositions*, and a posteriori propositions as empirical propositions. A *conceptual truth*, then, is just a true conceptual proposition.

Note that these definitions differ from Kant's. For Kant, representations are mind-independent entities of which we are immediately acquainted and with which we use to form judgments. Intuitions are particular representations that refer immediately

to an object, while concepts are general representations that refer to an object via their marks.

As one might expect, Bolzano found his definitions to be an improvement on Kant's:

Our distinctions have the advantage of being thoroughly objective ones, ones which do not rest upon certain external circumstances but rather are grounded in an internal attribute that depends on the object itself and which already pertains to the proposition in themselves, not merely to their appearances in the mind (to judgements and cognitions). (Přihonský 1850, pp. 32-33)

Thus, Bolzano's definitions purport to provide a better alternative to Kant's. Instead of focusing on mind-dependent phenomena, Bolzano thought that a more promising epistemology of mathematics could be given if one relied on mind-independent semantic entities and the relations that hold among them.

Bolzano's theory of propositions also shows why Kant's philosophy of mathematics gets off on the wrong foot. As mentioned, mathematical truths for Bolzano are *conceptual truths*. They concern truths that consist solely of concepts in Bolzano's sense. As such, they can only be grounded in further conceptual truths. Thus, as many commentators have concluded, intuition cannot ground mathematical knowledge and can be dispensed with by knowing a proposition's genuine ground.¹⁵

I argue that Bolzano's reasoning is problematic. The inference from the claim that mathematics consists of conceptual truths to the claim that one can and ought to dispense with intuition is too quick: it equivocates subjective with objective grounds—a distinction that Bolzano takes to be of central importance. By conflating subjective grounds with objective grounds, one misunderstands the role intuition plays in mathematical inference. For Kant, intuition was never meant to provide justification for a judgment or ground in Bolzano's sense. Intuition is not intended to play the role of X in "I perceive X , therefore I know Y " where Y is some mathematical proposition. Intuition is meant to underwrite immediate knowledge of axioms and inferences that went beyond traditional logic.

To think that Bolzano could appeal to the relation of ground-consequence as a way to show the untenability of Kant's doctrine of a priori construction is misguided and most likely based on a misunderstanding of the role that a priori construction plays in mathematical cognition and inference. In fact, one can see how Bolzano's ideal of objective proofs in mathematics is *compatible* with Kant's account. As Bolzano is well aware of, the ground of cognition can be radically different from the ground of truth.

Furthermore, an appeal to ground and consequence leaves unexplained how Bolzano would account for our knowledge of ground-consequence relations and ungroundable grounding propositions, i.e., axioms. As Kant has argued, knowledge of either outstrips the logical power of general logic and concept-containment relations.

According to Bolzano, axioms are composed of simple concepts, which he calls *axiomatic concepts*. One might think that Bolzano can explain our knowledge of ax-

¹⁵ See for example Lapointe (2011, p. 17) and Rusnock (1999, pp. 405-6).

ioms by appealing to our grasp of axiomatic concepts. However, even then, Bolzano would have to provide an explanation of which combinations of axiomatic concepts constitute a genuine ground and which do not. If the axiomatic concepts are truly simple, then it is not clear how they alone could disclose such information.

Resisting pure intuition at all costs, Bolzano offers an alternative account of the missing variable that underlies not only synthetic a priori judgments such as axioms but inferred cognition as well:

Nothing, I say, but that the understanding *has* and *knows* the two concepts *A* and *B*. I think that we must be in a position to judge about certain concepts merely because we have them. For, to say that somebody has certain concepts surely means that he knows them and can distinguish them. But to say that he knows them and can distinguish them means that he can claim something about one of them which he would not want to claim about the others [...] (Bolzano 1837, §305)

Without further details from Bolzano, it is not clear exactly how the “having” or “knowing” of concepts puts one in a cognitive position to go beyond their mere contents and form synthetic a priori judgments, and in particular, to know that a candidate for an axiom is a genuine axiom.

Even if Bolzano's theory of propositions is unable to provide a purely conceptual explanation of how we come to know axioms as grounds, there might still be good reason to think that Kant overestimated the extent to which intuition was needed to secure *mathematical inference*.

In both *WL* and *BD*, Bolzano claims that Kant goes wrong in thinking that without intuition, the only logical relationships we can discover between judgments are the ones given by Aristotelian syllogistic relationships and concept-containment relations. Indeed, as Bolzano emphatically remarks:

It seems to me therefore that one of Kant's literary sins was that he attempted to deprive us of a wholesome faith in the perfectibility of logic through an assertion very welcome to human indolence, namely, that logic is a science which has been complete and closed since the time of Aristotle. It seems to me that it would be much better to assert as a kind of practical postulate that faith in the perpetual perfectibility not only of logic but of all science should be maintained. (Bolzano 1837, §9)

Bolzano ambitiously sought to go beyond the limits on cognition set forth by the first *Critique*, attempting to redefine logic in such away that allowed one to uncover the objective dependence relations that held among the truths of mathematics without recourse to intuition.

The force of Bolzano's dispensability objection to Kant can then be salvaged by pointing out that, at least, inferences from known truths did not require the use of intuition. In other words, Bolzano's dispensability objection can be understood as an argument that Aristotle's logic was nowhere near exhaustive and that Kant had mistakenly underestimated our capacity to draw reliable inferences without intuition.

On this point, it is difficult not to side with Bolzano as the development of logic has arguably shown Aristotelian logic to be, in fact, not exhaustive. However, to assess whether Bolzano's dispensability argument is truly a knock-down argument against Kant, more needs to be said concerning why Bolzano took Aristotelian logic to be incomplete. To this, Bolzano simply says:

There are forms of inference which tell us how to derive certain propositions from others which cannot be derived from them by any syllogisms, no matter how often they are repeated. (Bolzano 1837, §262)

Bolzano has several examples in mind. For instance, Bolzano claims that the inference from “All *As* are *Bs*” and “All *Bs* are *As*” to “Whatever is *A* or is *B*, is *A* and *B*” cannot be reduced to a syllogism. One might be tempted to think that such inferences, if truly irreducible to syllogisms, would already refute Kant’s claim that Aristotle’s logic was complete. Hence, such inferences would open the possibility of synthetic inferences that did not require the use of intuition.

In *BD* and especially *WL*, Bolzano attempts to systematize various syllogistic and non-syllogistic forms of inference. While in *BD*, Bolzano admits that “on the other hand, how propositions with simple concepts could be proved other than through a syllogism, I really do not know” (Bolzano 1810, §20), he concedes that the inference from “this is a triangle” to “this is a figure the sum of whose angles equals two right angles” requires knowledge of extra-logical facts (i.e., the general fact that “the sum of the angles in any triangle equals two right angles”). Indeed, Bolzano remarks that a number of his inference rules are such that their “validity or invalidity . . . can be assessed only if we have knowledge of matters outside of logic” (Bolzano 1837, §223).

Recall that for Kant, general logic concerns the “logical form in the relation of cognitions to one another” (Kant 1781, B76-9). As I have pointed out, Kant’s characterization of general logic does not necessarily entail, as Kant thought, that the principles of logic are exhausted by Aristotle’s syllogisms. Regardless of what one takes as the principles of logic, what matters for Kant is that they concern the mere form of thinking. Thus, I take it that Kant’s critical project is not essentially tied to Aristotelian logic.¹⁶

Insofar as Bolzano’s proposed non-syllogistic inferences are sufficiently formal, Kant would not disagree with Bolzano. However, as mentioned before, many of Bolzano’s proposed inferences admittedly do not possess this feature and require the use of non-logical facts. To use the above example, the inference from “this is a triangle” to “this is a figure the sum of whose angles equals two right angles,” requires knowing the general fact that “the sum of the angles in any triangle equals two right angles.” A Kantian presumably has no problem in permitting forms of inference that presuppose knowledge of non-logical facts as counting as genuinely part of logic. In fact, he calls these inferences as being part of applied logic, which is distinct from general logic because the former concerns inferences of a particular subject matter. However, the point is that a truly logical inference involves the mere form of thinking without resorting to non-logical facts.

Neither does an appeal to Bolzano’s celebrated notion of deducibility (i.e., “Ableitbarkeit”) get at a viable alternative to Kant’s intuition in the case of securing mathematical inference (Bolzano 1837, §154). According to Bolzano, one can arrive at (some) reliable forms of inference by taking an argument, letting certain terms of

¹⁶ This is not to say that serious complications will not arise if Kant’s critical philosophy is adapted to a different set of logical laws.

the argument be variable, and testing to see if every substitution instance that makes the premises true makes the conclusion true. One might contend that such a method isolates certain forms of reliable inference, which can be used to eliminate intuition from the inference to a proposition from its ground. However, the problem with this strategy is that Bolzano still does not have the resources to explain cognition of grounds since not every instance of a deducibility relation obtaining between sentences, say, P and Q implies that there is a grounding relation obtaining between P and Q . In modern parlance, not every case of logical validity implies metaphysical grounding or explanation.¹⁷ Furthermore, Kant could argue that choice of constant and variable terms (unless one takes all non-logical terms as variable) is a choice that violates the formality of logic. Even if Bolzano rejects Kant's conception of logic as formal, Bolzano has to concede that in order to mechanically carry out a deducibility test, one has to have prior cognitive access to a potentially infinite class of truths, the cognition of which we either do not have as finite epistemic subjects or one of which remains in need of explanation.

Ultimately, Kant has at his disposal intuition to explain cognition of both ground-consequence relations and grounds. Such knowledge is arguably outside of the scope of Bolzano's objective definitions, and his notion of ground-consequence and deducibility. These resources alone are unable to provide a substitute for Kant's intuition, and thus, the dispensability objection loses its argumentative force.¹⁸

In sum, I hope to have shown that the particularity of intuitions does not preclude them from enjoying a necessary role in mathematical cognition. Furthermore, neither does an appeal to Bolzano's more "objective" definitions get at problematic features of Kant's account nor do they provide resources that can act as an alternative to Kant's account. Even if our knowledge of mathematics ought to follow from concepts without an appeal to intuition, Bolzano has yet to provide a compelling explanation as to how that is possible.

While various commentators have attributed Bolzano with undermining Kant's doctrine of pure intuition on the basis of mathematical definitions and new logical notions, I have argued that if Bolzano undermines Kant's doctrine of intuition, it is not Bolzano's alternative definitions nor his logical notions of grounding and deducibility that do the job.

¹⁷ One can give the following counterexample. Given any proposition P , P logically implies itself. However, not every proposition *grounds* itself.

¹⁸ At the end of the day, I argue that the tension between Kant's and Bolzano's respective definitions of terms such as "intuition" and "a priori" is merely apparent. I see no reason why a Kantian has to reject Bolzano's definitions as long as he could supplement them with her own. For Kant, a mathematical inference is composed of a grounding cognition and an a priori construction that allows for the possibility of genuine cognition from the ground to the inferred proposition in a way that goes beyond the laws of traditional logic. Kant's account leaves room for what Bolzano would consider to be an objective ground of a mathematical truth as its objective ground. As I see it, Kant's account does not necessarily impose prohibitions on mathematical practice in a way that would preclude any instances of a ground-consequence from being genuine cases of ground-consequence.

5 Concluding Remarks: A Deeper Disagreement

To conclude, I would like to make some remarks concerning why Bolzano felt generally dissatisfied with Kant's critical philosophy. I argue that many of Bolzano's objections stem from his rejection of the claim that a "critique of pure reason" is necessary for securing and explaining reliable forms of inference and claims to a priori knowledge. While Kant thought he was performing a service for philosophy, which simultaneously preserved and explained the a priori status of mathematics, Bolzano saw the conclusions of the critique as inhibiting mathematical progress. The following passage summarizes this sentiment:

The path therefore which the *Critique* pursues for deciding the question about the validity of our cognition is not at all the right one. The faculty of cognition of human beings or of thinking beings in general must not be analyzed first. Rather it is necessary to look into the nature of the truths in themselves [...] Only after this has happened can one usefully deal with cognizing and the conditions of cognizing. (Příhonský, 1850, p. 146)

Bolzano saw a critique of reason to be unnecessary because we can be sure of the reliability of our inferences and our claims to a priori cognition by studying general features of propositions in themselves. The nature of Bolzano's project reveals that he was not discouraged by Kant's conclusion that our theoretical synthetic a priori cognition cannot outstrip what is given to us.

While Kant or a Kantian would not necessarily disagree with Bolzano's view that there is an objective realm of propositions standing in ground-consequence relations, one of the central conclusions of the first *Critique* is that there are certain limitations on our ability to cognize such relations. Namely, our cognition is limited to objects of possible experience. From a Kantian point of view, because Bolzano seeks to extend the use of reason beyond possible experience, Bolzano's project runs the risk of problematically extending our claims of a priori cognition beyond reason's proper bounds, making reason vulnerable to the endless metaphysical disputes that motivated Kant's *Critique* in the first place. The naive set theory (which features an arguably inconsistent unrestricted comprehension axiom¹⁹) developed in Bolzano's 1851 *Paradoxien des Unendlichen*²⁰ might well be a perfect example of the unfortunate consequences that may result from reason extending its claims of cognition beyond its proper bounds.

If human agents with particular cognitive capabilities are the ones carrying out Bolzano's project as he envisions it, then it seems that any account regarding the secure advancement of knowledge needs to make some essential reference to those cognitive capabilities. Before we can engage in the type of foundational reform that Bolzano suggests, it seems necessary to gauge the extent to which we are epistemically and mechanically capable of tracking objective dependence relations at all.

¹⁹ See Bolzano (1851, §14).

²⁰ "Paradoxes of the Infinite."

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